Chapter 8

Motion on a Fractal

In earlier chapters we learned about the geometry of branching and fractal objects, *how* they grow, and *how* to measure their fractal dimension. But this is only part of the story. Lungs are shaped like *natural* or *random fractals* (structures that grow with an element of chance and over a range of magnifications have the same fractal dimension). Some rocks are also shaped like natural fractals. The brain, coral, mountains, bacterial colonies, and the edge of ripped paper towels can also have fractal shapes. But now our question is *why*?

Why should the lungs have a branching, perhaps fractal pattern? How does this pattern help the lungs to function better? How does its fractal pattern help coral to grow and thrive? And if mountains are fractal, what are the consequences for water flow? If rocks are fractal, what are the implications for oil or mineral recovery? If a rough surface is fractal, what does that mean for the artist who paints on it? or the hobbyist who runs an electric current through it? Can a subway system whose map is a fractal efficiently serve a city? Can a fractal pattern help things to work better?

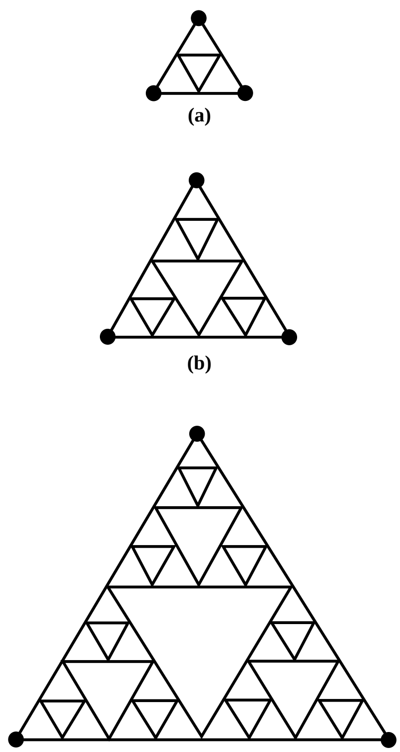
An important question for current research is: How do fractal structures improve function? On a practical level, engineers try to understand how water passes through soil and rock in the ground ("hydrological transport properties of porous rock"), the resistance to electrical current on rough surfaces (" electrical impedance of rough surfaces"), or why glass stays hot longer when it is heated than metal does ("thermal transport in amorphous materials").

How can we simulate a system created by random processes? Two natural fractals grown under the same conditions may have the same fractal dimension, but still be different from one another. No two electrodeposits or snowflakes or root systems or river deltas or lightning strokes or lungs or corals or termite tunnels or city subway systems are the same. So how can we compare members of each group among themselves (such as the branching deltas of two different rivers), and draw general conclusions? If each case is unique, how can we estimate how much oil we can recover from porous rock, how long it takes for a signal to travel through a nervous system, how Nature designs a coral colony for efficient feeding, or how well the human lung exchanges carbon dioxide and oxygen?

Q8.1: *Speculate:* Why are there similarities between the bronchial system, the circulatory system, and the nervous system? What is each attempting to accomplish? What is the task that must be carried out by each of these structures, and how do their branching structures help them carry out these tasks?

We start to analyze motion on a fractal by studying a simple system, one which retains the basic properties of a real system, but which can be easily manipulated and analyzed. This modeling process helps build our intuition about the behavior of real objects which have more complicated shapes.

The fractal shown in Figure 8.1 is called the Sierpinski gasket and has proved to be a workhorse for testing theories. Because of its simple network connections, many properties of this fractal, such as its fractal dimensions, can be easily found. Do you see that every intersection in the pattern—where two or more lines meet—is connected to four other intersections? Exception: the three external vertices (represented by dots in Figure 8.1), which are connected to only two other intersections in the pattern. Being connected to four other intersections is also true of intersections on a square grid.



(a) (b) (c)

Figure 8.1: Building the Sierpinski Gasket. Each step in the construction—(a), (b), (c)—uses three structures identical to that of the previous step. Only three steps are shown here. The same kind of steps are repeated indefinitely to create a "true" mathematical fractal.

> Q8.2: *Speculate:* Each intersection point on the Sierpinski gasket has four nearest neighbors. Then why is the Sierpinski gasket not just a rearrangement of a square grid?

In this chapter we begin by reconsidering an exercise we did earlier—finding out how many random steps it takes *on average* to travel a given distance from a starting point. Earlier we studied the random walker on a square grid, one in which each point had four connections to neighboring points (see Section 3.7 beginning on page 59). This was a two-dimensional grid. In this chapter we study what happens if instead you do a random walk between points on a *fractal* grid. Specifically, we perform a random walk on a Sierpinski gasket and find the average number of steps necessary to travel between two points on the gasket.

After studying the random walk on a fractal, we tackle another kind of problem: How does electricity flow on a fractal? This is analogous to asking, How does water flow through porous rock? or How efficiently will a railway system operate? or What kind of flight network should an airline design to maximize its profits?

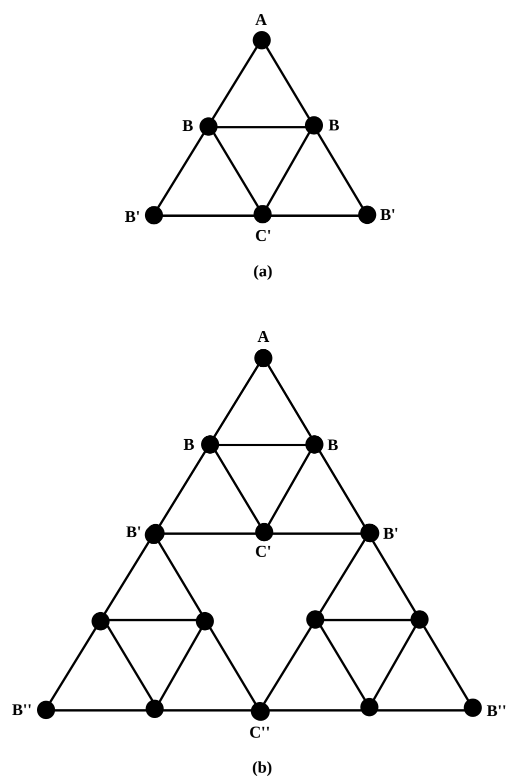


Figure 8.2: (a) Second step in building the Sierpinski gasket, with grid points labeled by dots (•). (b) Third step in building the Sierpinski gasket, with grid points also labeled by dots (•).

8.1 Random Walk on a Fractal

The *length of one step* in Figure 8.2 as the distance from point **A** to point **B**. Then the distance from point **A** to point **B″** is four steps. (From now on, we will call each intersection—where two or more lines meet—a **point** or a **grid point**.) *What do we mean by a random walk on the gasket?* If we are at point **A** or at either of the two point marked **B″** in Figure 8.2(b), then we are connected to only two other points on the gasket. A random step from **A** or **B″** will take us to one of the two nearest grid points, with a 0.5 probability of arriving at each point. If we are at any other point on the gasket, then we are connected to *four* other grid points. Then a random step will take us to one of these four nearest grid points, with a 0.25 probability of arriving at each point, just as in the case of the random walk on a square grid.

Q8.3: *Speculate:* If we start a random walker at Point **A**, how many steps will it take *on average* for it to reach Point **B″**? The points are separated by a distance of four steps. If this were a square grid, we might guess that the average number of steps necessary to travel this distance would be 16. For an explanation, see Section 3.5 beginning on page 52.

Q8.4: Do you think that it will take *on average* a greater or lesser number of steps than 16 to go from Point **A** to Point **B″**? Write down a brief argument to explain your prediction.

HandsOn 33: Random Walk on a Fractal

We now attempt to determine experimentally the average time it takes for a particle to move from one point to another on the Sierpinksi Gasket. The method is as follows: Roll a four-sided die, and move a random walker on the gasket. Keep count of how many steps the walker takes to move from one point to another point. By repeating this procedure, you will find the average time necessary to move specific distances.

Now for the details. "Time" in this case means the same thing as the "total number of steps" taken by the random walker. If it makes it easier for you, call the unit of time 1 second, and assume the walker takes 1 step per second. Then 10 steps take 10 seconds, 50 steps take 50 seconds, and so forth. The number of steps—the time—to go from one point to another will most likely be different for different trials because of the randomness of the process, based on the flip of a coin or the throw of a die. Hence we will try to predict the *average* times to go from one point to another. To emphasize that the times are averages, we use brackets $\langle\ \rangle$. For example, the average time it takes to go from Point **A** to Point **B** is written $\langle T_{AB} \rangle$.

In the following experiment we use a 4-sided die (available in some game stores) to produce random numbers. To use such a die, roll it and look at the number which appears along the bottom edge of the die. This is the result of your roll.

> Q8.5: If you have only an ordinary 6-sided die, use four of the faces. That is, if a 5 or 6 comes up, ignore it and roll again. *Speculate:* Is this really the same as using a 4-sided die?

To measure the average time it takes to go from one external vertex to another external point on the Sierpinski gasket (for example, from **A** to **B′** in Figure 8.2(a)), carry out the following steps. First, place your "walker" (e.g., penny, pen top, thumb tack) at point **A** on the gasket. Flip a coin. If the coin comes up heads, move the walker down to the right. If the result is tails, move the walker down to the left. Now there are four possible directions for the next step. Roll the die. Move the walker according to the diagram in Figure 8.3. Select from the three possible orientations the one that fits your current position, and move the walker accordingly.

1. Find the average time $\langle T_{AB'} \rangle$ (that is, the average number of random steps) required to move from point **A** to *either* of the two points labeled **B′** in Figure 8.2(a) or 8.2(b). Start with your

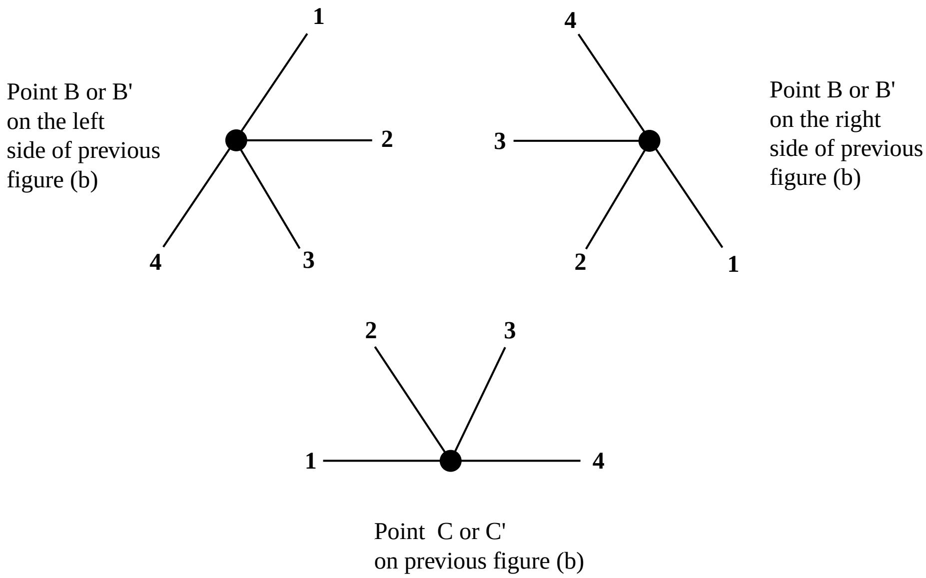


Figure 8.3: Choosing the direction for the next step from labeled points in Figure 8.2(b). In each case, the direction is given by the number resulting from the roll of a 4-sided die.

walker at point **A** and tally the number of steps it takes to arrive at either point labeled **B′**. Record this number and repeat this trial 10 times. On each trial, record also the number of steps needed for the walker to arrive for the first time at point **C′**. Record only the number of steps prior to the **first** arrival at **C′**. Ignore any later returns to **C′**. Then continue with your walk until you arrive at either **B′**. (It is possible that the walker will not reach **C′** on a given trial. In that case, you won't have data for arrival at **C′** for that trial. That's all right.) Record your results on a data sheet.

2. The goal of this second task is to evaluate the average time $\langle T_{AB''} \rangle$ (= average number of random steps) to move from Point **A** to either of the points **B″** in Figure 8.2(b). As in Task 1, for each trial start the walker at Point **A** and record the number of steps taken before arriving at either of the equivalent vertices labeled **B″**. Also record the number of steps necessary to first arrive at

either point $\mathbf{B}'$ on each trial (this data can be used to supplement your results in Task 1). Carry out 10 such trials recording your results for each one on a data sheet.

3. Now analyze your data. Because 10 is a small number of trials, your average results may be very different from your neighbor's. You can obtain better results by averaging the results of your trials with those of the other members of the class.

Q8.6: Compute $\langle T_{AB'} \rangle$, the average time it takes to move from point **A** to either point $\mathbf{B}'$. Use the data from Task 1. Using the data from Task 2, compute $\langle T_{AB''} \rangle$. How do these two averages compare to what the movement would be like on a square grid?

Q8.7: Compute $\langle T_{C'B'} \rangle$. (Hint: You counted the number of steps necessary to arrive first at $\mathbf{C}'$ and then to arrive at either $\mathbf{B}'$. For each trial where you passed through $\mathbf{C}'$, subtract the former from the latter to find the number of steps needed to move from $\mathbf{C}'$ to either $\mathbf{B}'$.

Q8.8: From Figure 8.2(a), see if you can justify the following equation:

$$\langle T_{AB'} \rangle = 1 + \langle T_{BB'} \rangle. \tag{8.1}$$

Here we have written 1 for $\langle T_{AB} \rangle$, the time needed to go from A to *either* point B in Figure 8.2(a). Use this equation to compute $\langle T_{BB'} \rangle$ from your data. (With two more such equations, it is possible to solve exactly for the average first arrival time at an external vertex on a Sierpinski gasket.

END ACTIVITY

8.2 The Power Law for a Random Walker on a Sierpinski Gasket

In earlier chapters you studied the properties of a random walk along a line, and on a two-dimensional square grid. We found that for a random walk along a straight line or on a flat surface, after N steps (or a "time of N") the mean square distance traveled $\langle R^2 \rangle$ is proportional to N, or equivalently, the root mean square displacement (see Section 3.7 beginning on page 59) is

$$\sqrt{\langle R^2 \rangle} = N^{1/2}. \tag{8.2}$$

On a fractal, interconnections are not so regular as on a straight line or square lattice. Therefore the relationship between the number of steps and the average distance covered may not be the same. We try to describe this difference by a change in the exponent 1/2 on N in Eq. 8.2 to a different value, as yet undetermined, which we shall call s:

$$\sqrt{\langle R^2 \rangle} = N^{s}. \tag{8.3}$$

For the Sierpinski gasket we can derive the exponent s by using a procedure similar to the following: Suppose that in the experiment of HandsOn 33 on page 193, your data showed

$$\langle T_{AB'} \rangle = 5\langle T_{AB} \rangle. \tag{8.4}$$

Values of T (the time) in Eq. 8.4 mean "number of steps," the same as N in Eq. 8.3. Then Eq. 8.4 and your experiments tell us that to go twice the distance (that is, to go from **A** to **B′** which is twice the distance from **A** to **B**), it takes on average 5 times longer. For this case, Eq. 8.3 can be written as:

$$2 = 5^{s}. \tag{8.5}$$

Use the properties of logarithms to show that

$$\log 2 = \log(5^{s}) = s \log 5 \tag{8.6}$$

or

$$s = \frac{\log 2}{\log 5} = 0.431. \tag{8.7}$$

This exponent is less than the 1/2 for a random walk on a square grid as shown in Eq. 8.2 above. This result would indicate that movement on a Sierpinski gasket differs from that on a square grid. This movement in fact has been named *anomalous diffusion*.

8.3 Diffusion on Sierpinski Gasket

In the preceding section we learned that the average distance a random walker moves in a given length of time is determined by the connectivity of the points on the grid on which the walker moves. Although we studied a simple model, a random walk on a Sierpinski gasket, the results illustrate the behavior of random walkers in materials with complicated connections. Remember that the phenomenon of diffusion is explained using the model of random walkers. In fact, you have just studied diffusion on a fractal, in this case diffusion on the Sierpinski gasket.

In the following section we study the electrical resistance of the Sierpinski gasket, and how it varies with size. The experiments to be performed, and the behavior of the Sierpinski gasket made of resistors, is analogous to the behavior of the Sierpinski gasket made of pipes through which water flows. So if you have not studied electrical circuits yet, you can think of circuits as systems of water pipes.

HandsOn 34: Resistance of a Fractal Network

Recall how one calculates the net resistance of two resistors in series or in parallel. We offer two different ways to think of resistance: one is electrical resistance, the other is resistance to the flow of water through pipes.

In Figure 8.4(a), two resistors are in series in a circuit with a battery. Resistors literally *resist* the flow of current. When one resistor follows another, the total resistance of the circuit is $R_1 + R_2$.

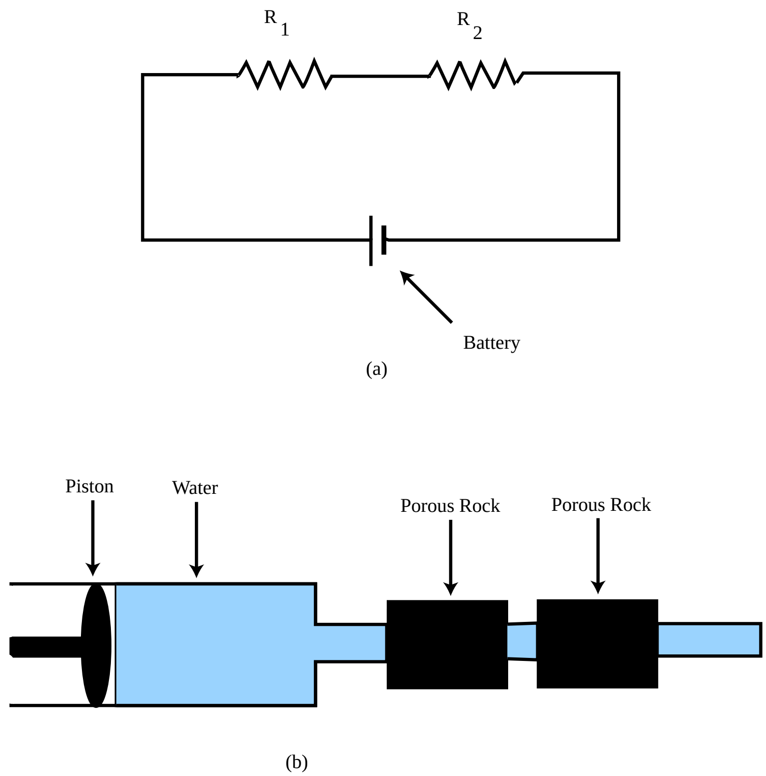


Figure 8.4: (a) Two resistors in series with a battery. The resistance of this circuit is the sum of the resistances of the two resistors. (b) Two porous rocks in series with a piston pump that drives water through them. The resistance to the flow of the water is the sum of the resistances of each of the rocks to water flow.

Figure 8.4(b) shows a piston applying pressure to drive water through two porous rocks connected in series with pipes between them. If a constant force is applied to the piston, then the rate of flow of water through the porous rocks is determined by their resistances. Since the two rocks are in a row, the resistances of the two porous rocks are added together.

What is important about series resistances is that they add. If you double the length of a resistor, its resistance doubles. To describe how resistors add in series, we could say that the resistance increases with the total length of the resistors. So, for resistors in series, the total resistance is proportional to length, L, i.e., $R \sim L$.

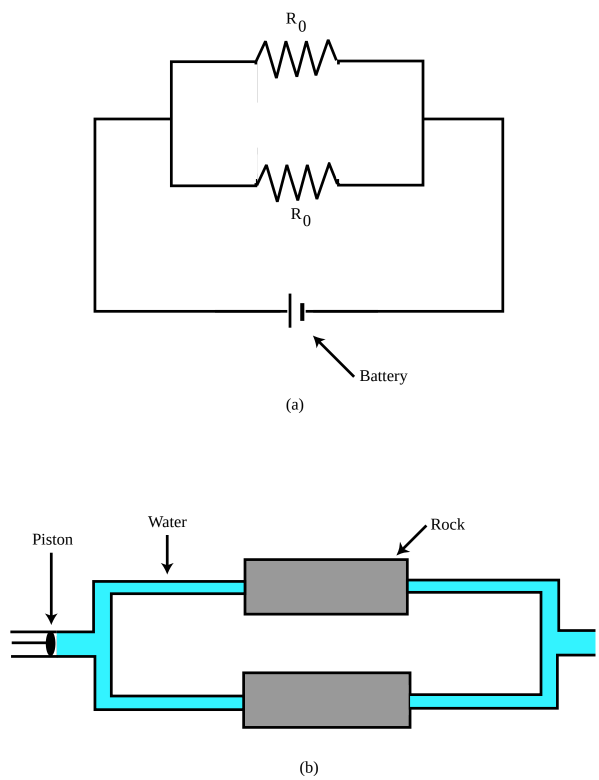


Figure 8.5: (a) Two equal resistors in parallel with a battery. The resistance of this circuit is half that of the same circuit with a single resistor. (b) Two porous rocks in parallel in a water pipe circuit. The two rocks are identical. The resistance to the flow of water is half the resistance of the pipe circuit with one porous rock.

By contrast, Figures 8.5(a) and 8.5(b), show two resistors and two porous rocks in parallel. The piston applies the same force to each of the porous rocks. Each rock experiences the same water force as it would if it were alone, with no parallel rock alongside it. Each rock permits the passage of as much water as if it were alone. Taken together then, the two rocks pass *twice* as much water as either rock alone.

Twice as much water for the same force on the piston means that the two rocks in parallel have *half* the resistance to water flow as one rock alone. What is important is that the total resistance *decreases* as more resistors are placed in parallel. If two equal resistors are placed in parallel, their combined resistance is half of the resistance of either one alone.

We could also reason that placing resistors in parallel is equivalent to increasing the cross-sectional area A through which current can flow. Since the resistance decreases with more resistors in parallel, and hence greater cross-sectional area, we can say that $R \sim \frac{1}{A}$.

Combining our result for series and parallel resistors, we conclude that for ordinary objects the dependence of resistance R of material of cross-section A and length L is

$$R = \rho \frac{L}{A}. \tag{8.8}$$

Here the constant of proportionality ρ is called the **resistivity**. The resistivity ρ has a different value for different materials. For example, copper and carbon have different values of resistivity.

END ACTIVITY

What does this all have to do with fractal dimension?

Let's apply the logic of the preceding section to objects with different dimensions, as shown in Figures 8.6(a),(b), and (c). The wire in Figure 8.6(a) is effectively a one-dimensional object, because we vary only its length and not its radius.

> Q8.9: Does the actual value of the radius matter in determining whether an object can be considered one-dimensional? Can you imagine cases when it does, and other cases when it does not?

In particular, if we double the length of such a wire, its resistance doubles. From this we conclude that for a one-dimensional object, $R \sim L = L^1$ (here we emphasize that the exponent is 1).

Now consider the two-dimensional square sheet resistor in Figure 8.6(b). We treat it as two-dimensional because throughout the following we hold its thickness t constant.

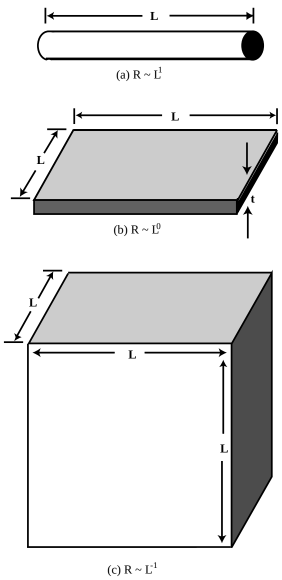


Figure 8.6: With each shape the dependence of the resistance on the length is indicated. (a) One-dimensional resistor: if we double its length the total resistance doubles. (b) Two-dimensional resistor: if we double both the length and the width of the sheet (while holding the thickness constant), its resistance stays constant. (c) Three-dimensional resistor: if we double the length of each side of the cube, its resistance is cut in half.

Q8.10: When does such a sheet behave like a two-dimensional object, and when is its third dimension important?

The cross-sectional area of the sheet is $A = tL$. So, if we double its length L *and* double its width L, then the resistance remains constant since L cancels out in Eq. 8.8):

$$R = \rho \frac{L}{A} = \rho \frac{L}{tL} = \frac{\rho}{t}. \tag{8.9}$$

The result is that in two dimensions the resistance does not depend on the size of the sheet if we increase *both* the length *and* width by the same factor (while keeping the thickness constant). We write this formally as $R \sim L^0$. Recall that any quantity to the zero power has the value unity (unity = 1), so in this case R is constant.

> Q8.11: *Speculate:* How is it possible that the resistance is independent of the length and width of the sheet in two-dimensions? Physically what does this mean?

Let's consider the change of resistance of a sheet as we increase its width and length separately. If we double its length in the direction of current flow, then the resistance of the sheet doubles. This action is equivalent to treating the sheet as a one-dimensional object. In terms of water flow through the sheet, we have doubled the distance over which the water must be driven, and hence doubled the resistance to the flow of water.

On the other hand, if we double the width of the sheet in Figure 8.6(b), this is equivalent to adding an identical resistor in parallel. This cuts the resistance in half. Or, in terms of water flow, we have doubled the quantity of water that can flow through when the same force is applied to the piston, which means the resistance to fluid flow has been cut in half.

This combination of series and parallel resistance changes yields the surprising result that in two dimensions, the resistance of a square sheet (of a given thickness) is not dependent on the edge length L of the sheet, but only on the material (that is, on the value of its resistivity ρ). Equivalently, for flow through a slab of uniform two-dimensional rock, the resistance is not dependent on the width or length of the slab!

> Q8.12: Is the above result true only for square two-dimensional sheets? If you double both the width and the length of a rectangular sheet, does the resistance remain the same?

Finally, let's consider the resistance of a three-dimensional cube as in Figure 8.6(c). If we double the cube's length in the direction of fluid or electric flow (making it no longer a cube but something called "a rectangular parallelepiped"), this doubles its resistance. If we double its width (equivalent to putting two of the new parallelepipeds in parallel), we halve its resistance. Finally, if we double the height of the cube, equivalent to putting four more parallelepipeds in parallel, we halve the resistance again. The result is a new cube with resistance

$$2 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$$

times the original resistance. From this we conclude that the resistance of a cube varies with the length of a side as $R \sim 1/L = L^{-1}$. This also follows directly from Eq. 8.8, since $A = L^2$ results in

$$R = \rho\frac{L}{A} = \rho\frac{L}{L^2} = \frac{\rho}{L}.$$

We have found that in one dimension, $R \sim L^{+1}$, in two dimensions, $R \sim L^0$, and in three dimensions, $R \sim L^{-1}$.

All of the above results regarding resistance in 1-, 2-, and 3-dimensional objects (with L describing the size of the object), can be summarized by writing

$$R \sim L^{2-d}. \tag{8.10}$$

where d is the dimension of the object. Verify this by substituting $d = 1$, 2, and 3 into Eq. 8.10 and comparing the results with the results of the previous analysis.

Q8.13: *Speculate:* Suppose that a mythical hypercube (a cube in four dimensions—don't even *try* to draw one!) has edge length L, and resistance R_0. What would its resistance be, as a multiple of R_0, if you doubled the edge length in all four dimensions.

Now we are experts on the how resistance behaves for objects with integer dimensions. We have attempted to understand the behavior of such objects using simple arguments about resistors in series and parallel.

But what about objects which have a more complicated geometry than that described by integer dimensions? What about fractal objects with non-integer dimensions? For example, what is the resistance of the Sierpinski gasket measured between a variety of points on its structure?

HandsOn 35: Measuring the Resistance of the Sierpinski Gasket

In the following experiment, you will assemble a fractal resistor network that has the structure of the Sierpinski gasket. In order to assemble a large gasket, this experiment requires cooperation among many collaborators or class participants.

In a classroom setting, each student is provided with nine identical resistors (e.g., 1 kΩ is fine) and a 5 cm × 5 cm circuit board (insulating fiberboard with holes punched in it) on which to mount the resistors. Each student group should be provided with an ohmmeter to measure electrical resistance.

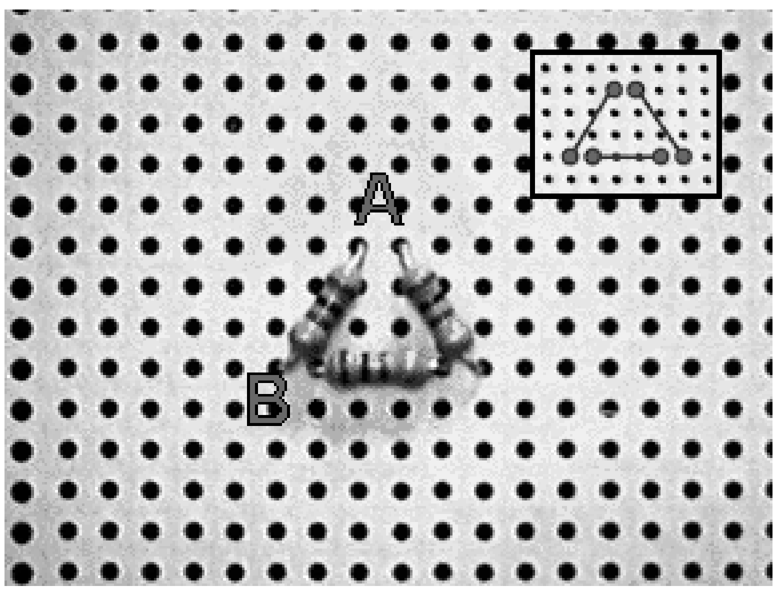


Figure 8.7: Construction of the Sierpinski Gasket resistor network following the generation process of Figure 8.1. Here the first generation is shown. Each step in the construction—shown on the following two figures—uses three structures identical to that of the previous step. The same kind of steps are repeated indefinitely to create a "true" mathematical fractal.

The steps in this experiment for groups of three students are:

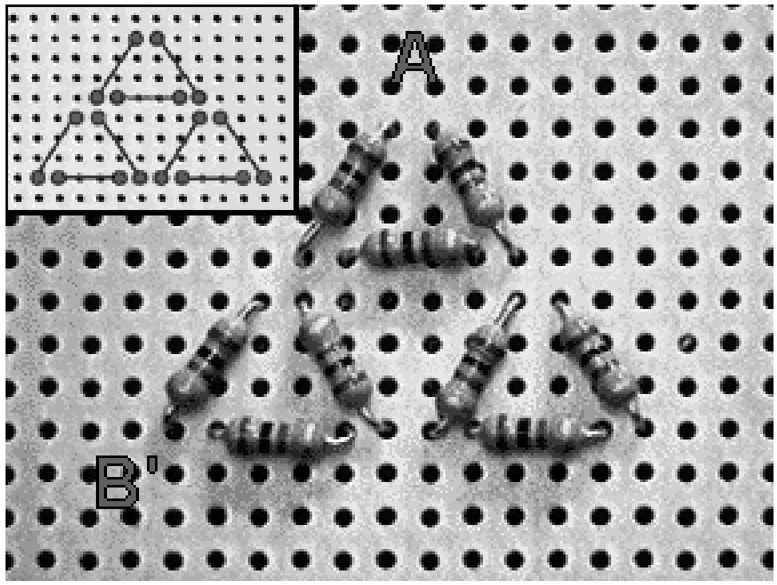


Figure 8.8: The first generation is repeated 3 times to create a second generation gasket with 9 resistors.

1. Each student of each team should assemble a 3-resistor gasket (a first generation Sierpinski gasket) as shown in Figure 8.7, then measure the resistance between points **A** and **B**. Record this resistance. Then each student should continue, assembling a 9 resistor second generation gasket as shown in Figure 8.8, measure the resistance between points **A** and **B′**, and record this resistance.

2. Each team of students should construct the circuit in Figure 8.9 by linking together the three circuits they made in Step 1. Measure the resistance between points **A** and **B″**, and record this value. *All teams should now compare resistance measurements to be sure that none of the assembled circuits are defective.*

3. Three teams should join their circuits together to build the next generation of the gasket shown in Figure 8.10. Measure the resistance between the outer vertices **A** and **B‴**, and record the value.

4. If the class is large enough (27 students), the process can be

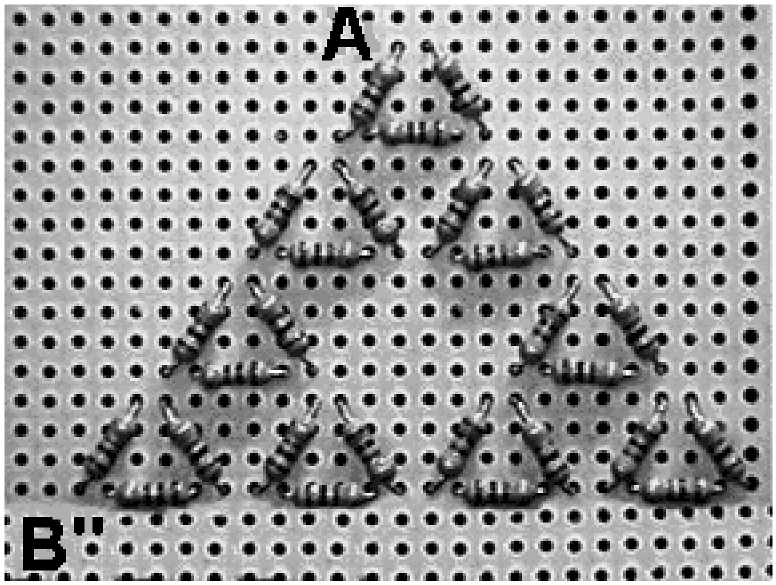


Figure 8.9: The second generation is repeated 3 times to create a third generation gasket with 27 resistors.

repeated one more time to produce another generation of the gasket. Again the resistance should be measured between the external vertices, and recorded. (If the class is smaller than 27, perhaps some students can build extra circuits.)

5. To analyze the data, plot on log-log paper the measured resistance from vertex to vertex along the vertical axis *vs.* the number of resistors along one side of the network (i.e., the vertex-to-vertex distance) along the horizontal axis. Draw the best straight line you can through these graphed points.

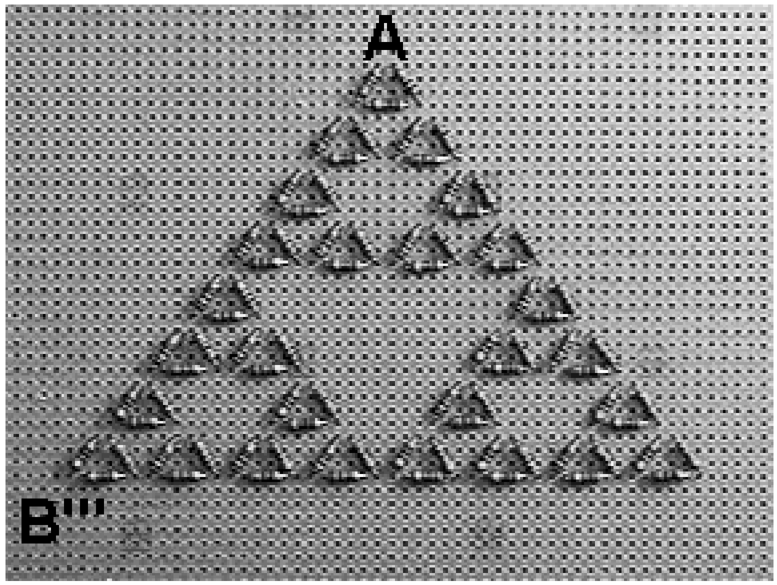


Figure 8.10: The fourth generation of the gasket with 81 resistors.

Q8.14: What would the slope be if you plotted on log-log paper the resistance of a one-dimensional chain of resistors as a function of the number of resistors? What would the slope be if you plotted on log-log paper the resistance of a series of conducting squares of different sizes as a function of the length L of their edges? And for a cube?

Q8.15: The slope you obtained in step 5 is a measure of the dependence of the electrical resistance on the size of the Sierpinski gasket circuit. Does the value you find make sense in relation to the resistance of objects of dimensions 1, 2 and 3 shown in Figure 7.6?

END ACTIVITY

8.3.1 Computing the Resistance of the Sierpinski Gasket

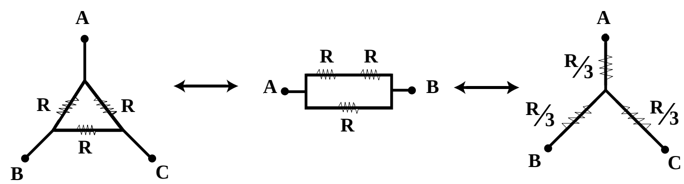


Figure 8.11: *The basic star-triangle transformation.* The triangle of resistors of resistance R_0 (1) is rearranged as a series of two resistors in parallel with a third resistor (2). We find the resistance between points A and B to be $2R_0/3$. We then replace the triangle of resistors with a star of resistors connecting points A, B, and C (3). Each of the star resistors is $R_0/3$.

We have made a *measurement* of the resistance of the Sierpinski gasket resistor network. How can we *calculate* a number with which to compare it? To compute the resistance of the Sierpinski gasket resistor network in Figures 8.7 to 8.10, we cannot use Eq. 8.8—there is no direct relationship to cross-sectional area or length. Instead, we use a method for analyzing networks of resistors called the **triangle-star method**. We apply the triangle-star method to Figure 8.7, as is indicated schematically in Figure 8.11(a).

Begin by analyzing the triangular connection of wires between vertices **A, B**, and **C** in Figure 8.11(a). We want to know the value of resistance between vertices **A** and **B**. We can redraw this circuit as two series resistors in parallel with the third resistor.

Each of the individual resistors still has a resistance R_o. If you know how to compute the resistance for a parallel resistor circuit, then you can solve this problem mathematically. The resistance between points **A** and **B** is $2R_o/3$.

You can also arrive at this result by a physical argument. Return to the analogy of pressure being applied to porous rocks. Suppose the current of water through one porous rock is I_o. If we apply the same pressure across two such rocks in series, the total current will be $I_o/2$ since the resistance is doubled (Figure 8.4).

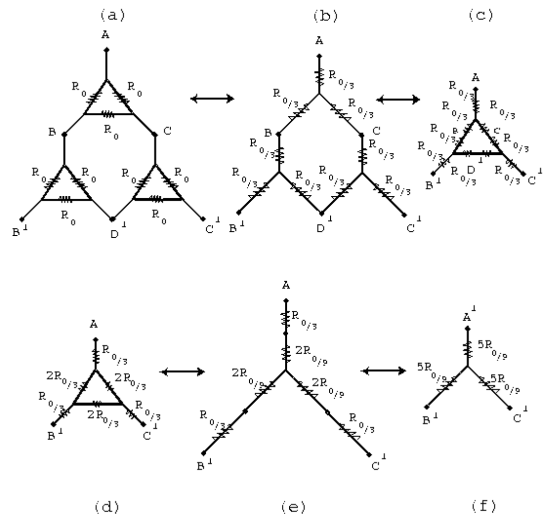


Figure 8.12: Application of the star-triangle transformation of Figure 7.8 to the gasket shown in Figure 8.2(a). In going from (a) to (b) in the figure above, each of the triangles is replaced by an equivalent star. A rearrangement of the internal resistors in (c) and (d) reveals that we have recovered another triangle with resistors of $2R_0/3$ on each leg. This triangle is in turn replaced with a star in (e) where each leg has a resistance od $2R_0/9$. Finally, adding the two resistances in each leg in (f), we are left with a star with each leg's resistance $5R_0/9$. The resistance between points A' and B' is then $10R_0/9$.

Now connect the two ends of a single porous rock across the outer ends of the two porous rocks in series, giving a result that looks like the middle diagram in Figure 8.11. Then apply the same pressure as before between the two ends of this structure (A and B). The same applied pressure forces the same current $I_o/2$ through the branch containing two rocks in series. In addition, a current I_o passes through the branch with the single porous rock. The sum of these two flows is $3I_o/2$. In brief, the same pressure as before causes a *greater* current (3/2 times as great) through the structure. To achieve this, the equivalent resistance for this circuit of porous rocks must be *smaller* than that of the single porous rock. How much smaller?

- Twice the current would mean half the resistance.
- Half the current would mean twice the resistance.
- So 3/2 the current means 2/3 the resistance.
- We say that the resistance and current are *inversely proportional* to one another.

Whether by mathematical formula or physical argument, we come to the conclusion that the resistance between points **A** and **B** is $2R_o/3$.

The next step of the argument is key: when analyzing a network, the resistors between points **A** and **B**, for example, can be replaced by another set of resistors *provided the new set of resistors produce the same resistance between these points.* For example, in the case of Figure 7.8(a), we can replace the "triangle" of resistors with a "star" of resistors each with resistance $R_0/3$. This preserves the resistances between the external points: the resistance between points **A** and **B** remains $2R_0/3$ after the substitution of the new arrangement of resistors. The resistance between point **A** and **C** also remains $2R_0/3$ after the substitution.

Having simplified the network in Figure 8.7, we apply this result to higher generations of the gasket. The triangle-star rearrangement of the circuit in Figure 8.11 is shown in Figure 8.12, which has five steps. The first step is to replace the three triangles of Figure 8.12(a) by stars in the same fashion as in Figure 8.11. The result yields one large internal triangle in step (c) of Figure 8.12. This in turn can be replaced by a star, again using the method of Figure 8.11. Proceeding in this fashion, we reduce any gasket of resistors to one equivalent star of known resistance. The results for the resistance between the external vertices for gaskets of increasing generation number are:

$2R_0/3$ for the initial triangle as in Figure 8.11;

$10R_0/9$ for the first step gasket (9 resistors) as in Figures 8.8 and 8.12;

$50R_0/27$ for the second step gasket (27 resistors) shown in Figure 8.9;

$250R_0/81$ for the third step (81 resistors) shown in Figure 8.10;

$1250R_0/243$ for the fourth step (243 resistors);

$6250R_0/729$ for the fifth step (729 resistors).

Generally for the n-th step ($3^{(n+1)}$ resistors) the resistance between two external vertices is:

$$\frac{2 \times 5^n}{3^{n+1}} R_0. \tag{8.11}$$

This is the general formula for the resistance expressed as a function of the transformation step number. But the basic question remains: if you pick up a gasket of length L, and then another of length L', what is the relationship between the two resistances?

What we are after is a relationship between the resistance and the size; here size is measured by the vertex-to-vertex length of the gasket. To derive this relationship, we *postulate* that as a function of the vertex-to-vertex distance L the resistance between vertices can be written as

$$R(L) = kL^q. \tag{8.12}$$

Here k is a constant of proportionality, and the exponent q tells us about the way resistance depends on size of the circuit [compare this equation with Eq. 8.10].

Having made this postulate, we test if it is valid. Compare the resistance for the original single triangle, $(2/3)R_0$ (Figure 8.11), with that of the first-step triangle, $(10/9)R_0$ (Figure 8.12). See that

$$R(2L) = \frac{5}{3} R(L). \tag{8.13}$$

If we substitute Eq. 8.12 into Eq. 8.13 we obtain

$$R(2L) = k(2L)^q = \frac{5}{3} R(L) = \frac{5}{3} kL^q. \tag{8.14}$$

Equate the second entry $k(2L)^q$ in Eq. 8.14 to the last entry $(5/3)kL^q$. This yields $2^q = 5/3$, so the exponent q is

$$q = \frac{\log(5/3)}{\log 2} = 0.737. \tag{8.15}$$

> Q8.16: Does this result of 0.737 make sense? The Sierpinski gasket has a dimension of 1.58, between $d = 1$ and $d = 2$. What is the resistance exponent for $d = 1$? What is it for $d = 2$? Do you see a pattern?

This method of star-triangle transformation to analyze the Sierpinski gasket can be generalized to analyze an arbitrary random network on a square or triangular grid. The power of this method is not restricted to this one example.

8.4 An Exact Solution for the Dimension of a Random Walk on a Sierpinski Gasket (Advanced)

We can derive the exponent s in Eq. 8.3 directly by using equations similar to Eq. 8.1. Begin by applying a test to $\langle T_{C'B'}\rangle$. Let us imagine that we execute four separate trials for randomly walking from $\mathbf{C}'$ to $\mathbf{B}'$. From $\mathbf{C}'$ there are four paths to $\mathbf{B}'$. Refer to Figure 8.2(a). On average, we expect that: on one trial we go directly to the leftmost $\mathbf{B}'$ in time $\langle T_{AB}\rangle$ (the same as the time to go between the adjacent point A and B); on one trial we go directly to the rightmost $\mathbf{B}'$; on one trial we go to the leftmost point B in time $\langle T_{AB}\rangle$, and then take time $\langle T_{BB'}\rangle$ to arrive at a $\mathbf{B}'$; and, finally, on average one of the walks will first take us the rightmost point $\mathbf{B}$ in time $\langle T_{AB}\rangle$, and from there it will take $\langle T_{BB'}\rangle$ to arrive at a $\mathbf{B}'$.

Express this analysis as an equation:

$4\langle T_{C'B'}\rangle$ = the average length of time to execute four trials from $\mathbf{C}'$ to $\mathbf{B}'$

$$
\begin{aligned}
&= \langle T_{AB}\rangle(\text{the time to go directly to the leftmost } \mathbf{B}') \\
&+ \langle T_{AB}\rangle(\text{the time to go directly to the rightmost } \mathbf{B}') \\
&+ \langle T_{AB}\rangle(\text{the time to go from } \mathbf{C}' \text{ to leftmost } \mathbf{B} \text{ (i.e., 1 step)}) \\
&+ \langle T_{BB'}\rangle(\text{the average time to get to } \mathbf{B}' \text{ from } \mathbf{B}) \\
&+ \langle T_{AB}\rangle(\text{the time to go from } \mathbf{C}' \text{ to rightmost } \mathbf{B} \text{ (i.e., 1 step)}) \\
&+ \langle T_{BB'}\rangle(\text{the average time to get to } \mathbf{B}' \text{ from } \mathbf{B}) \\
&= 4\langle T_{AB}\rangle + 2\langle T_{BB'}\rangle. \qquad (8.16)
\end{aligned}
$$

In short, our second equation relating the average internal times is:

$$\langle T_{C'B'}\rangle = \langle T_{AB}\rangle + \frac{\langle T_{BB'}\rangle}{2}. \qquad (8.17)$$

Finally, we apply the same logic to four trial random walks which are initiated at a point **B**. On average: one walk will take time $\langle T_{AB}\rangle$ to arrive at the other point **B**, and then the average time $\langle T_{BB'}\rangle$ to arrive at one of the points $\mathbf{B}'$; one walk will take time $\langle T_{AB}\rangle$ to move to point **A**, and then time $\langle T_{AB'}\rangle$ to arrive at $\mathbf{B}'$; one walk will take time $\langle T_{AB}\rangle$ to arrive at $\mathbf{C}'$, and then time $\langle T_{C'B'}\rangle$ to arrive at a $\mathbf{B}'$; and, one walk go directly to $\mathbf{B}'$ in time $\langle T_{AB}\rangle$. Summing these four ways to go, we have (with the parenthesized terms in order of the above description):

$$
\begin{aligned}
4\langle T_{BB'}\rangle &= (\langle T_{AB}\rangle + \langle T_{BB'}\rangle) + (\langle T_{AB}\rangle + \langle T_{AB'}\rangle) \\
&+ (\langle T_{AB}\rangle + \langle T_{C'B'}\rangle) + (\langle T_{AB}\rangle). \qquad (8.18)
\end{aligned}
$$

Simplifying this equation:

$$3\langle T_{BB'}\rangle = 4\langle T_{AB}\rangle + \langle T_{AB'}\rangle + \langle T_{C'B'}\rangle. \qquad (8.19)$$

Q8.17: Do your averaged quantities computed in Task 1 in HandsOn 33 on page 193 satisfy these equations? Check the numbers. Do you get better agreement using average times as found averaging over the data of all students?

Q8.18: Equations 8.1, 8.17, and 8.19 are a system of three simultaneous equations in the three unknowns $\langle T_{BB'} \rangle$, $\langle T_{AB'} \rangle$, and $\langle T_{C'B'} \rangle$ in terms of the known $\langle T_{AB} \rangle$. Try solving these equations to prove Eq. 8.4.